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RESEARCH ARTICLES

Functional Classical Mechanics and Rational Numbers

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Abstract—The notion of microscopic state of the system at a given moment of time as a point in the phase space as well as a notion of trajectory is widely used in classical mechanics. However, it does not have an immediate physical meaning, since arbitrary real numbers are unobservable. This notion leads to the known paradoxes, such as the irreversibility problem. A "functional" formulation of classical mechanics is suggested. The physical meaning is attached in this formulation not to an individual trajectory but only to a "beam" of trajectories, or the distribution function on phase space. The fundamental equation of the microscopic dynamics in the functional approach is not the Newton equation but the Liouville equation for the distribution function of the single particle. The Newton equation in this approach appears as an approximate equation describing the dynamics of the average values and there are corrections to the Newton trajectories. We give a construction of probability density function starting from the directly observable quantities, i.e., the results of measurements, which are rational numbers.

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1. INTRODUCTION

The conventional widely used concept of the microscopic state of the system in classical Newtonian mechanics [1] at some moment of time as the point in phase space, as well as the notion of trajectory and the microscopic equations of motion have no direct physical meaning, since arbitrary real numbers are unobservable. Observable physical quantities are only presented by rational numbers [2–4], see also the discussion of concepts of space and time in [2–8].

In [9] it was suggested a "functional" formulation of classical mechanics. The fundamental equation of the microscopic dynamics in the functional approach is not the Newton equation, but the Liouville equation for the distribution function of a single particle. The Newton equation in this approach appears as an approximate equation describing the dynamics of the average values, and there are corrections to the Newton trajectories. The functional formulation of classical mechanics gives also an approach to the solution of the irreversibility problem.

In this note we give a construction of the probability density function starting from the directly observable quantities, i.e., the results of measurements, which are rational numbers.

2. STATES AND OBSERVABLES IN FUNCTIONAL CLASSICAL MECHANICS

Usually in classical mechanics the motion of a point body is described by a trajectory in the phase space, i.e., the values of the position and momentum as functions of time, which are solutions of the equations of Newton or Hamilton.

However, this mathematical model is an idealization of the physical process, rather far separated from reality. Every physical body has the spatial dimensions, such a mathematical point gives only an

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approximate description of the physical body. The mathematical notion of a trajectory does not have a direct physical meaning, since it uses arbitrary real numbers, i.e., infinite decimal expansions, while the observation is only possible, in the best case, in rational numbers, and even among them only with some error. Therefore, in the "functional" approach to classical mechanics, we are not starting from the Newton equation, but with the Liouville equation.

Consider the motion of a classical particle along a straight line in the potential field. The general case of many particles in the 3-dimensional space is discussed below. Let (q, p) be coordinates on the plane $\mathbb{R}^2$ (phase space), $t \in \mathbb{R}$ is time. The state of a classical particle at time t will be described by a function $\rho = \rho(q, p, t)$, it is the density of the probability that the particle at time t has the position q and momentum p.

The description of a mechanical system with the help of probability distribution function $\rho = \rho(q, p, t)$ does not necessarily mean that we are dealing with a set of identically prepared ensemble of particles. Usually in probability theory one considers an ensemble of events and a sample space. But we can use the description with the function $\rho = \rho(q, p, t)$ also for individual bodies, such as planets in astronomy (the phase space in this case the 6-dimensional). In this case one can think on the "ensemble" of different astronomers which observe the planet, or on the "ensemble" of different models of behavior of a given object for one "intelligent" observer. Actually, it is implicitly always dealt with the function $\rho = \rho(q, p, t)$ which takes into account the inherent uncertainty in the position and momentum of the body.

The specific type of function ρ depends on the method of the preparation of the state of the classical particle at the initial time and the type of potential field. When $\rho = \rho(q, p, t)$ has sharp peaks at $q = q_0$ and $p = p_0$, we say that the particle has the approximate values of the position and momentum q_0 and p_0.

Emphasize that the exact determination of the position and momentum can not be done not only in quantum mechanics, where there is the Heisenberg uncertainty relation, but also in classical mechanics. Always there are some errors in setting the position and momentum. There are classical uncertainty relations:

$$\Delta q \Delta p > 0,$$

i.e., the uncertainty (errors of observation) in the determination of the position and momentum is always positive (nonzero). The concept of arbitrary real numbers, given by the infinite decimal series, is a mathematical idealization, such numbers cannot be measured in the experiment.

Therefore, in the functional approach to classical mechanics the concept of precise trajectory of a particle is absent, the fundamental concept is a distribution function $\rho = \rho(q, p, t)$, and δ-function as a distribution function is not allowed.

We assume that the continuously differentiable and integrable function $\rho = \rho(q, p, t)$ satisfies the conditions:

$$\rho \geq 0, \quad \int_{\mathbb{R}^2} \rho(q, p, t) dq dp = 1, \ t \in \mathbb{R}\,. \tag{2.1}$$

If $f = f(q, p)$ is a function on the phase space, the average value of f at time t is given by the integral

$$\overline{f}(t) = \int f(q, p) \rho(q, p, t) dq dp\,. \tag{2.2}$$

In a sense, we are dealing with a random process $\xi(t)$ with values in the phase space. Motion of a point body along a straight line in the potential field will be described by the equation

$$\frac{\partial \rho}{\partial t} = -\frac{p}{m} \frac{\partial \rho}{\partial q} + \frac{\partial V(q)}{\partial q} \frac{\partial \rho}{\partial p}\,. \tag{2.3}$$

Here $V(q)$ is a potential field and $m > 0$ is the mass of the body.

Equation (2.3) looks like the Liouville equation, which is used in statistical physics to describe a gas of particles, but here we use it to describe a single particle.

The characteristics equations for (2.3) are Hamilton's equations

$$\dot{q} = \frac{\partial H}{\partial p}, \ \dot{p} = -\frac{\partial H}{\partial q}\,, \tag{2.4}$$

where the Hamiltonian is

$$H = \frac{p^2}{2m} + V(q)\,. \tag{2.5}$$

Emphasize again that the Hamilton equations (2.4) in the current functional approach to mechanics do not describe directly the motion of particles, but they are only the characteristics equations for the Liouville equation (2.3).

If the distribution $\rho_0(q,p)$ for $t=0$ is known, we can consider the Cauchy problem for the equation (2.3):

$$\rho|_{t=0} = \rho_0(q,p)\,. \tag{2.6}$$

Consider the case when the initial distribution has the Gaussian form:

$$\rho_0(q,p) = \frac{1}{\pi ab} e^{-\frac{(q-q_0)^2}{a^2}} e^{-\frac{(p-p_0)^2}{b^2}}\,. \tag{2.7}$$

At sufficiently small values of the parameters $a>0$ and $b>0$ the particle has the position and momentum close to the q_0 and p_0. For this distribution the average values of the position and momentum are

$$\overline{q} = \int q\rho_0(q,p)dqdp = q_0\,,\ \ \overline{p} = \int p\rho_0(q,p)dqdp = p_0\,, \tag{2.8}$$

and the dispersion

$$\Delta q^2 = \overline{(q-\overline{q})^2} = \frac{1}{2}a^2,\ \ \Delta p^2 = \overline{(p-\overline{p})^2} = \frac{1}{2}b^2\,. \tag{2.9}$$

For the free motion ($V=0$) we get

$$\overline{q}(t) = q_0 + \frac{p_0}{m}t\,,\ \ \overline{p}(t) = p_0\,, \tag{2.10}$$

and the dispersion increases with time:

$$\Delta q^2(t) = \frac{1}{2}(a^2 + \frac{b^2t^2}{m^2}). \tag{2.11}$$

Consider the case of the free motion of the particle, when $V=0$. In this case the equation (2.3) has the form

$$\frac{\partial\rho}{\partial t} = -\frac{p}{m}\frac{\partial\rho}{\partial q} \tag{2.12}$$

and the solution of the Cauchy problem is

$$\rho(q,p,t) = \rho_0(q - \frac{p}{m}t, p)\,. \tag{2.13}$$

Using expressions (2.7), (2.13),

$$\rho(q,p,t) = \frac{1}{\pi ab}\exp\{-\frac{(q-q_0-\frac{p}{m}t)^2}{a^2} - \frac{(p-p_0)^2}{b^2}\}\,, \tag{2.14}$$

we get the time dependent distribution of coordinates:

$$\rho_c(q,t) = \int \rho(q,p,t)dp = \frac{1}{\sqrt{\pi}\sqrt{a^2+\frac{b^2t^2}{m^2}}}\exp\{-\frac{(q-q_0-\frac{p_0}{m}t)^2}{(a^2+\frac{b^2t^2}{m^2})}\}\,, \tag{2.15}$$

while the distribution of momenta is

$$\rho_m(p,t) = \int \rho(q,p,t)dq = \frac{1}{\sqrt{\pi}b}e^{-\frac{(p-p_0)^2}{b^2}}. \tag{2.16}$$

Thus, for the free particle the distribution of the particle momentum with the passage of time does not change, and the distribution of the coordinates changes. There is, as one says in quantum mechanics, the spreading of the wave packet. From (2.15) it follows that the dispersion Δq^2 increases with time:

$$\Delta q^2(t) = \frac{1}{2}(a^2 + \frac{b^2 t^2}{m^2}) . \tag{2.17}$$

Even if the particle was arbitrarily well localized (a^2 is arbitrarily small) at $t = 0$, then at sufficiently large times t the localization of the particle becomes meaningless, there is a *delocalization* of the particle which accounts for irreversibility.

Corrections to the Newton's trajectories for the nonlinear coupling are computed in [9].

3. PROBABILITY DENSITY FUNCTION AND RATIONAL NUMBERS

The probability density function is real-valued. However, our initial point was that the real numbers are unobservable. Does the use of real-valued probability density function as a fundamental notion of mechanics contradict to our initial thesis? In this section we construct the probability density function (formula (3.3)) starting from the directly observable quantities using the methods of mathematical statistics. An important point is that the probability might be a real number and this is admissible, since the probabilities are not directly observable.

The directly observable quantities are the results of measurements and they are rational numbers. Consider a measurement of an observable $\mathbb{X}$. For simplicity we consider a one-dimensional observable, the generalization is simple. Every measurement device has an error, which must be taken into account. Roughly speaking, the measurement errors can be divided into two types: systematic and random errors [10]. If we perform repeated measurements, the systematic error does not change and random error changes randomly (we do not consider the part of the error that changes regularly, because it can be excluded by the statistical methods). It is natural to model the random error by a random variable. There is no theory of systematic error ("In fact, the only theory of systematic errors is that they must be identified and reduced until they are much less than the required precision... However, this goal is often not attainable" [10]). Therefore, although the systematic error is constant, our ignorance of the systematic error is also modelled by a random variable, because of the absence of another theory.

Thus, the result of a measurement is a random variable X, it is rational-valued. Moreover, since the precision (sensitivity) of every instrument is finite, X takes values not on the whole field of rational numbers $\mathbb{Q}$, but rather on the lattice, $X \in \frac{\nu}{\mu}\mathbb{Z}$, where the rational fraction $\frac{\nu}{\mu}$ is the measuring sensitivity of the instrument and $\mathbb{Z}$ are integers. So, the probabilities $p_m = \Pr[X = \frac{\nu}{\mu}m]$, $m \in \mathbb{Z}$, are defined. The probabilities p_m are real and this is admissible, since the probabilities are not directly observable. They can be considered as a limit of relative frequencies:

$$p_m = \lim_{n\to\infty} \frac{n_m}{n} \quad \text{in probability, i.e.,} \quad \lim_{n\to\infty} \Pr\left[\frac{n_m}{n} - p_m\right] = 0,$$

where n is the number of experiments and n_m is the number of experiments where the realization of X is equal to $\frac{\nu}{\mu}m$. This is the law of large numbers [11]. The fact that probabilities are real numbers, actually, is not surprising, since the limit of rational sequences is not necessarily a rational number.

Now consider the dynamics. Let us measure the observable $\mathbb{X}$ once again at some moment of time $t > 0$. We want to predict the probabilities of the results of this measurement on the condition that we know the result of the measurement at time $t = 0$. If we describe a state as a sum of delta functions and solve the Liouville equation with such initial conditions (this is equivalent to Newton's equation), we will get incorrect predictions. For example, consider the free motion on the real line. Assume that at time $t = 0$ we obtained that momentum is equal to zero with the precision allowed by our instrument. Then we can conclude that at any time $t > 0$ the particle still will be in its initial position. But in general this is not true, since the momentum can be very small (smaller than our measuring sensitivity), but not zero. In this case, if t is large enough, the position of the particle can be changed considerably (see (2.11)).

Thus, in order to take the growth of the error with the time into account, we must consider the states as continuous distributions. Let us assign some continuous real-valued random variable $\widetilde{X}$ to

our discrete random variable X. Let $\widetilde{X}$ be distributed according to some probability density function $\rho_*(x)$ which satisfies the condition

$$p_m = \int_{\frac{\nu}{\mu}(m-\frac{1}{2})}^{\frac{\nu}{\mu}(m+\frac{1}{2})} \rho_*(x)dx. \tag{3.1}$$

We assume that $\widetilde{X}$ is normally distributed:

$$\rho_*(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{(x-x^*)^2}{2\sigma^2}}, \tag{3.2}$$

where x^* is a mean value (which can, but not necessarily, be referred as a "true" value of the observable $\mathbb{X}$) and σ^2 is a dispersion.

$$\sigma^2 = \sigma^2_{\text{syst}} + \sigma^2_{\text{rand}},$$

where σ^2_{syst} and σ^2_{rand} are the summands that correspond to the systematic and random error accordingly. In fact, the further discussion does not depend critically on the form of distribution. We made an assumption about normal distribution for simplicity, but there are also some physical and mathematical reasons to choose this distributions among others.

Again, the notion of real-valued probability density function ρ_* does not contradict to the thesis that real values are unobservable, because the probability density function is not an observable. This is an abstract, theoretical object, which is useful, because we can approximate the relative frequencies using the notion of real-valued probability density function:

$$\frac{k\{X \in [a,b]\}}{n} \approx \int_a^b \rho_*(x)\, dx.$$

Here $k\{X \in [a,b]\}$ is the number of experiments where the realization of X belongs to $[a,b]$ (for example, $a, b \in \mathbb{Q}$) and n is the general number of experiments (it is assumed that n is large).

Usually we do not now the expectation value x^* and the dispersion of random error σ^2_{rand} (and hence, we do not know the probability density function ρ_*, we only assume that it has the form (3.2) with unknown parameters), but rather we have to estimate them using the methods of mathematical statistics. The dispersion of systematic error σ^2_{syst} is assumed to be known from the measuring instrument certificate. Let $X^{(1)}, \dots, X^{(n)}$ be n copies of $\widetilde{X}$, i.e., independent and identically distributed (according to the probability density function ρ_*) random variables (the results of n measurements). Then the following formulas are used to estimate the expectation and dispersion of the random error:

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X^{(i)}, \quad S^2_{\text{rand}} = \frac{1}{n-1}\sum_{i=1}^{n}(X^{(i)} - \overline{X})^2.$$

It is well-known that random variable $\overline{X}$ is normally distributed with the same expectation x^* as every of $X^{(1)}, \dots, X^{(n)}$. The dispersion of the random error in the estimation of $\overline{X}$ is reduced by n times and is equal to σ^2_{rand}/n. Hence, its estimation is S^2_{rand}/n [11]. The dispersion of the systematic error does not depend on the number of measurements and still equals to σ^2_{syst} (this is not a rigorous conclusion, since there is no theory of systematic error and the use of the formalism of random variables is not very correct for this). Therefore, the dispersion of the general error of the estimation of x^* is

$$S^2 = \frac{S^2_{\text{rand}}}{n} + \sigma^2_{\text{syst}}.$$

Now we construct the probability density function:

$$\rho_n(x) = \frac{1}{\sqrt{2\pi S^2}}\, e^{-\frac{(x-\overline{X})^2}{2S^2}}. \tag{3.3}$$

If n is large (in practice, $n > 30$ is enough), then $\rho_n(x)\Delta x$ has the meaning of the probability for the mean value x^* to belong to the interval $(x - \Delta x, x + \Delta x)$.

Remark 1. The last assertion can be understood by physicists, but is not completely correct from the mathematical point of view. Since x^* is not a random variable, the probability for it to belong to the definite interval is either zero or one. More rigorous formulation of the assertion is the following: $(1/\sqrt{2\pi})e^{-x^2/2}\Delta x$ is approximately the probability for $\frac{\overline{X}-x^*}{\sqrt{\frac{S^2}{n}}}$ to belong to the interval $(-\Delta x, \Delta x)$.

Note that, in fact, ρ_n is a random function, because it depends on the random variables $\overline{X}$ and S^2.

If $n \to \infty$, then $\overline{X} \to x^*$, $S^2_{\text{rand}}/n \to 0$, $S^2 \to \sigma^2_{\text{syst}}$ in probability. Denote

$$\rho_\infty(x) = \frac{1}{\sqrt{2\pi\sigma^2_{\text{syst}}}}\, e^{-\frac{(x-\overline{X})^2}{\sigma^2_{\text{syst}}}}.$$

This is also a random function. The following proposition holds:

Proposition 1.

$$\lim_{n\to\infty} \Pr\{X^{(n)} \in [a,b]\} = \int_a^b \rho_\infty(x)dx$$

in probability, i.e.,

$$\lim_{n\to\infty} \Pr\left\{\Pr\{X^{(n)} \in [a,b]\} - \int_a^b \rho_\infty(x)dx\right\} = 0,$$

if $a = \frac{\nu}{\mu}(m - \frac{1}{2})$, $b = \frac{\nu}{\mu}(l - \frac{1}{2})$ *for some* $m, l \in \mathbb{Z}$ *(in other words,* $a, b \in \frac{\nu}{\mu}\mathbb{Z} + \frac{1}{2}$*).*

This is a corollary of the limit theorems of probability theory (the law of large numbers and the central limit theorem) and condition (3.1).

If we perform the repeated measurements, we can predict the probabilities of the results of the next measurement in the limit $n \to \infty$ using the constructed probability distribution function (3.3). This justifies the use of the described construction.

4. CONCLUSIONS

It is shown that the use of real-valued probability density function as a fundamental concept of functional classical mechanics does not contradict to the thesis that the real irrational numbers are unobservable, since the density function is not a directly observable value. The construction of the probability density function based on the rational-valued results of measurements and an argumentation for this construction are given. It would be interesting to extend these results to the case of quantum mechanics, see [12].

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