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STOCHASTIC PROPERTIES OF DEGENERATED QUANTUM SYSTEMS

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We study Schrödinger equation with degenerated symmetric but not self-adjoint Hamiltonian. The above properties of the quantum Hamiltonian arise in the description of the asymptotic behavior of the regularizing self-adjoint Hamiltonians sequence. A quantum dynamical semigroup corresponding to degenerated Hamiltonian is defined by means of the passage to the limit for the sequence of the regularizing dynamical semigroups. These semigroups are generated by the regularizing self-adjoint Hamiltonians. The necessary and sufficient conditions are obtained for the convergence of the regularizing semigroups sequence. The description of the divergent sequence of semigroups requires the extension of the stochastic process concept. We extend the stochastic process concept onto the family of measurable functions defined on the space endowed with finite additive measure. The above extension makes it possible to describe the structure of the partial limits set of the regularizing semigroups sequence.

Keywords: Open quantum system; finitely additive measure; semigroup.

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1. Introduction

In this paper we study Cauchy problem for Schrödinger equation:

$$i\frac{d}{dt}u(t) = \mathbf{L}u(t), \quad t \in (0,+\infty) = \mathbb{R}_+, \tag{1.1}$$

$$u(+0) = u_0 \in H, \quad H = L_2(\mathbb{R}^d), \tag{1.2}$$

where Hamiltonian $\mathbf{L}$ is symmetric but not self-adjoint linear operator in Hilbert space $H = L_2(R^d), d \in \mathbb{N}$.

According to the axiomatics of quantum mechanics (Ref. 5, Sec. 2.3.2), the observable of the quantum system in Hilbert space H is an arbitrary linear self-adjoint operator in the space H. Let $B(H)$ be Banach space of bounded linear operators in the space H equipped by the operator norm. Any linear continuous non-negative normalized functional on the space $B(H)$ is called the state of the quantum system. Any unit vector ψ of the space H is called the vector of the state of the quantum system. The quantum state ρ_ψ determining by the rule ρ_ψ : $\rho_\psi(\mathbf{A}) = (\psi, \mathbf{A}\psi), \mathbf{A} \in B(H)$, is said to be the pure state corresponding to the unit vector ψ. We denote by Σ the set of all quantum states.

The dynamics of vectors of the state of the quantum system is defined by some linear self-adjoint operator (Hamiltonian) $\mathbf{L}$ in the space H according to the rule $u_0 \to u(t) = \mathbf{U_L}(t)u_0$, where $\mathbf{U_L}(t) = e^{-i\mathbf{L}t}$ is the unitary group in the space H with the generator $\mathbf{L}$. Let $B^*(H)$ be the conjugate Banach space for Banach space $B(H)$. The self-adjoint Hamiltonian $\mathbf{L}$ generates the semigroup $T_t, t > 0$, of maps of the space $B^*(H)$ into itself which acts on some state $\rho \in B^*(H)$ by the rule $\rho \to T_t(\rho)$, where $T_t\rho(\mathbf{A}) = \rho(\mathbf{U_L}(t)^*\mathbf{A}\mathbf{U_L}(t))$ for any operator $\mathbf{A} \in B(H)$. This semigroup $T_t, t > 0$, is called the dynamical semigroup of transformations of the set of quantum states Σ generating by the Hamiltonian $\mathbf{L}$.

The loss of the self-adjointness of the operator $\mathbf{L}$ contradicts the quantum mechanics postulates (see Ref. 23). The ill-posed Cauchy problem (1.1)–(1.2) with the symmetric degenerated Hamiltonian without self-adjointness is considered instead of the sequence of Cauchy problems with self-adjoint Hamiltonians containing a small parameter before the highest derivatives. This problem is similar to the ill-posed Cauchy problem for the first-order partial differential equations admitting discontinuous solutions of shock wave class and the loss of the uniqueness property (Hamilton–Jacobi equation, the system of hydrodynamic equations (see Refs. 16, 18 and 20)). In fact the above problems are considered instead of the sequence of well-posed problems with small viscosity parameter before the highest derivatives. In both cases the ill-posed problem is the simplified scheme of the sequence of well-posed problems. In these cases the vanishing viscosity method is used. This method needs the application of topological analysis of the convergence to the approximating solution sequence. Both ill-posed problems admit the loss of the uniqueness property. Therefore the problem of studying the partial limits set of the approximating solutions sequence arises here. This paper contains the investigation of above problems for the linear ill-posed Cauchy problem for degenerated Schrödinger equation.

In this paper we assume that $\mathbf{L}$ is the second-order linear differential operator given by the following symmetric differential expression

$$\mathbf{L}v(x) = \frac{\partial}{\partial x}\left(g(x)\frac{\partial}{\partial x}v(x)\right) + \frac{i}{2}\left(a(x)\frac{\partial}{\partial x}v(x) + \frac{\partial}{\partial x}(a(x)v(x))\right) \tag{1.3}$$

on the corresponding maximal domain. Here the functions $g(x)$ and $a(x)$ are real-valued and measurable, the function $g(x)$ is non-negative.

In the physical sense, function $g(x)$ gives the dependence of the mass of the quantum system on its position, and function $a(x)$ is the vector potential of the electromagnetic field. Schrödinger equation (1.1) with Hamiltonian (1.3) describes the dynamics of the quantum system with the position-dependent mass.

The carriers of the charge in the semiconductor whose mass $m(x)$ is defined by the properties of the semiconductor matter give the example of the above system. Therefore the mass $m(x)$ depends on the point of the coordinate space (see Ref. 11). The dynamics of carriers of the charge can be described by Schrödinger equation with Hamiltonian (1.3), where the coefficient $g(x)$ and the mass of the system are connected by the relation $g(x) = \frac{h}{2m(x)}$ (here h is the Planck constant). If the function $g(x)$ is uniformly bounded away from zero, then the operator $\mathbf{L}$ is uniformly elliptic self-adjoint operator in the space H. If the function $g(x)$ vanishes on some set of the coordinate space then the operator $\mathbf{L}$ can be symmetric but not self-adjoint operator in H. In the last case the operator $\mathbf{L}$ cannot be the Hamiltonian of a quantum system according to the postulates of quantum mechanics. Cauchy problem for Schrödinger equation (1.1)–(1.3) with degenerated Hamiltonian is the simplified description of the asymptotic behavior of quantum systems when particle masses goes to infinity.

The union of quasiparticles (known as phonons) arising in the model of a lattice vibration gives the other example of physical systems with position-dependent mass. The dependence of fonon's energy on its momentum has a different character in different matters. In the framework of the acoustic law the energy of a phonon is the function of its momentum similar to linear one. In the framework of the optical law the energy of phonon depends on its momentum similarly as quadratic function. Let the lattice consists of two matters such that the energy of the vibration of some frequency obeys the optical law in the first matter and the acoustic law in the second one. Then the quantum Hamiltonian of such quasiparticle is given by formula (1.3), where function $g(x)$ is positive in the domain of the first type matter, and $g(x)$ equals 0 in the domain of the second type matter.

The third example is supplied by Lagrange system such that Hessian of Lagrange function vanishes on some nontrivial manifold of the phase space (this manifold is called the singular one). Lagrangian of electrodynamics or weak-relativistic Lagrangian gives the example of Lagrangian with nontrivial singular manifold (see Ref. 19). Cauchy problem for Lagrange–Euler equation for Lagrange system with the nontrivial singular manifold is usually ill-posed. In this work we investigate the model example of Lagrange system with one-dimensional coordinate space R. The mass of this system is given by the formula $m(x) = 1$, if $x \in \Omega$, and $m(x) = h$, if $x \in R\backslash\Omega$, where Ω is some subset of the coordinate space. Then for each $h > 0$ dynamics of the state vectors of the model system is defined by Cauchy problem (1.1)–(1.3) with self-adjoint Hamiltonian $\mathbf{L}_h$. In this case the asymptotic behavior of dynamics of the system in the semiclassical limit, when Planck constant h tends to 0, is defined by Cauchy problem (1.1)–(1.3) with symmetric degenerated Hamiltonian $\mathbf{L}$. Thus, Hamiltonian $\mathbf{L}$ with the above properties is inadmissible by axiomatics of quantum mechanics, but this Hamiltonian describes the limit model.

Typical properties of Cauchy problem with degenerated Hamiltonian are presented by the following two model problems.

Problem (I) is Cauchy problem (1.1)–(1.3) with $g(x) = \chi_\Delta(x)$, $a(x) = b\chi_{\Delta_1}(x)$, where b is real parameter, $\chi_B(x)$ is the characteristic function of the set $B \subset R$, $\Delta_1 = (0, +\infty)$ and $\Delta = R\backslash\Delta_1$ (see Ref. 25).

Problem (II) is Cauchy problem (1.1)–(1.3) with $g(x) = \chi_\Delta(x)$, $a(x) = \sum_{i=1}^n b_i(x)\chi_{\Delta_i}(x)$, where $\Delta_i, 1 \le i \le n$, is the union of pairwise disjoint closed segments of the real line $\mathbb{R}$, $\Delta = \mathbb{R}\backslash(\bigcup_{i=1}^n \Delta_i)$, and the functions $b_i(x) \in L_\infty(\Delta_i)$ have generalized derivatives $b_i' \in L_\infty(\Delta_i)$ (see Ref. 27).

The incorrectness of Cauchy problem (1.1)–(1.3) is the corollary of both degeneration of operator $\mathbf{L}$, and the loss of the smoothness of the operator coefficients on the closure of its degeneration set.

This paper has the aim to find those properties of problem (1.1)–(1.3) which are stable with respect to taking down degeneration as the particular case of the stability with respect to a small perturbation of coefficients of the problem. Therefore we investigate ill-posed Cauchy problem (1.1)–(1.3) with the help of the regularization method. This method is based on the consideration of the generalized sequence of Cauchy problems with initial condition (1.2) for the regularized equations

$$i\frac{d}{dt}u(t) = \mathbf{L}_\varepsilon u(t), \quad t \in \mathbb{R}_+, u(+0) = u_0, \quad \varepsilon \in E, \tag{1.4}$$

where the set of regularizing parameters E is some directed set endowed with the partial order $\prec$. The directed set of regularizing operators $\mathbf{L}_\varepsilon, \varepsilon \in E$, approximates the operator $\mathbf{L}$. In this paper we consider the following examples: E is the set of positive integers $\mathbf{N}$ with the natural order; E is the subset of positive real numbers with limit point $e^* = 0$ endowed with the inverse order; E is the neighborhood of the origin in some Hilbert space which is partially ordered by decreasing of its elements norm.

For example, the directed set of operators $\mathbf{L}_\varepsilon, \varepsilon \in (0,1), \varepsilon \to +0$, given by (1.3) with functions $g_\varepsilon(x) = g(x) + \varepsilon$ instead of $g(x)$, approximates the operator $\mathbf{L}$ for the problem (I).

2. The Aims and Preliminary Results

Since the rigorous formulations of the main results require the additional consideration we give the preliminary results here.

If dynamics of quantum states vectors is defined by the self-adjoint Hamiltonian then it is reversible. However, the consideration of some quantum mechanics problems (such as open quantum systems, quantum stochastic processes and quantum measurements (see Refs. 1, 29 and 17)) leads to the necessity of the studying of irreversible transformations of the quantum states union. On the other side, the irreversible transformations of the quantum states union arise in the investigations of asymptotic behavior of quantum systems dynamics depending on the parameter. The irreversible transformations of this type arise in the passage to the limit in the sequence of quantum states dynamics with position dependent mass when the parameter of Planck constant tends to zero (see

Ref. 25). In this article the irreversible transformation of the quantum states set Σ generating by the symmetric nonself-adjoint Hamiltonian is investigated as the limit element of the set of reversible transformations of Σ generating by the self-adjoint Hamiltonians. We study the convergence and generalized convergence of the sequences of transformations of the set Σ in the topology of weak* convergence.

The main aims of our investigation are:

(i) to obtain the transformation of the quantum states set which is defined by the regularization of Cauchy problem (1.1)–(1.3) as the limit of the regularizing transformations sequence;
(ii) to study the qualitative properties of the obtained transformation.

Since the symmetric operator $\mathbf{L}$ is not the generator of a contractive semigroup in the space H, we investigate some neighborhood of the operator $\mathbf{L}$ in the space of linear operators $\mathcal{L}(H)$.

We say that the sequence $\{\mathbf{L}_n\}$ of self-adjoint operators in the space H is the regularization of the symmetric operator $\mathbf{L}$ if it converges to some dilatation of operator $\mathbf{L}$ in the strong graph topology (see Ref. 21 and Definition 4.1 below). Cauchy problem with the initial data (1.2) and the generating operator $\mathbf{L}_n$ is called the regularizing Cauchy problem. In this work the convergence of the sequence of regularizing Cauchy problems solutions $\{u_n(t) = \mathbf{U}_{\mathbf{L}_n}(t)u_0, n \in \mathbb{N}\}$ is studied in the weak and strong topologies of the space H; in addition, the convergence of the sequence $\{\rho_{u_n(t)}\}$ of corresponding regularizing density operators is studied in the topology of weak$*$ convergence of the quantum states space $B^*(H)$.

In general case, if Hamiltonian of the problem (1.1)–(1.3) is not the maximal dissipative operator, then there is no convergence of the sequence of regularizing state vectors in the space H, and there is no convergence of the sequence of regularizing density operators in weak$*$ topology of the space $B^*(H)$. The paper, Ref. 27, includes the conditions which are sufficient or necessary for the convergence of the sequence of regularizing solutions in the strong and weak topologies of the space H. In Ref. 26, the divergence of any sequence of regularizing density operators in weak$*$ topology of the space $B^*(H)$ is proved. Also the set of all possible particular limits of numerical sequences of values of regularizing density operators on the points of the space $B(H)$ is investigated in Ref. 26.

If Hamiltonian $\mathbf{L}$ of Cauchy problem (1.1)–(1.3) is self-adjoint, then it generates the unitary dynamics of quantum states vectors and defines the dynamical semigroup on the quantum states set in the space $B^*(H)$. In this work the dynamics of the quantum states set connected with the regularization of Cauchy problem (1.1)–(1.3) is determined as the following set-valued map $\Phi(t)$:

$\Phi(t)$ maps semiaxe R_+ into the set of subsets of the space $B^*(H)$ such that for any positive real t and any element $\mathbf{A} \in B(H)$ the numerical set of values of functionals from $\Phi(t)$ on the element $\mathbf{A}$ coincides with the set of particular limits of

the numerical sequence $\{(u_n(t), \mathbf{A}u_n(t))\}$ *of values of regularizing density operators* $\rho_{u_n(t)}$ *at time* t *on the element* $\mathbf{A}$.

This dynamics introducing by means of regularization loses the properties of single-valuedness because each particular limit of the numerical sequence $\{(u_n(t), \mathbf{A}u_n(t))\}$ pretends the role of one of the possible realizations of the dynamics of the mean value of operator $\mathbf{A}$ generating by Cauchy problem (1.1)–(1.3).

For obtaining the information on the stochastic properties of the regularization, we consider the directed set E of regularization parameters as the measurable space with some non-negative normalized measure defining on the algebra 2^E of all subsets of the set E such that the measure of any bounded subset of directed set E equals zero (Refs. 12, 26 and Sec. 5.2 below). Each measure μ with these properties induces the measure on the particular limits set of the numerical sequence $\{(u_\varepsilon(t), \mathbf{A}u_\varepsilon(t))\}$ of values of regularizing density operators on the arbitrary element $\mathbf{A}$ of the space $B(H)$. Then we say that the regularization of Cauchy problem (1.1)–(1.3) defines the stochastic process on the measurable space $(E, 2^E, \mu)$ with values in the set of quantum states Σ. According to the definition of stochastic process the measure μ must be countable additive. But we are interested only in the limit behavior of the regularizing sequence. Then we consider only measures μ supported in arbitrary vicinity of limit point $\varepsilon = 0$. Therefore we extend the concept of stochastic process by means of the reduction of countable additivity requirement to finite additivity requirement for measure μ. Then the above stochastic process induces the measure on the sets $\Phi(t), t > 0$, of values of the set-valued map. Our definition of stochastic process gives opportunity to describe the structure of values of this map.

This view on the regularization procedure treats the divergence of the sequence of regularizing problems solutions as the presentation of stochastic properties of the considered ill-posed problem but not the loss of the physical sense of this problem. Note that the nonuniqueness of generalized solutions obtaining by the regularization procedure is the characteristic property of these problems as the nonlinear boundary value problems (see Ref. 22) and the linear boundary value problem with degenerated operators (see Refs. 32, 25 and 27) or operators with discontinuous coefficients in nondivergent form (see Ref. 24). In Refs. 2 and 31 the stochastic approach to the study of the solutions set of some nonlinear Cauchy problem (for Vlasov equations or hydrodynamics equations) is developed by introducing the measure on the solutions set.

One of the aims of this paper is the investigation of the above described stochastic process with the values in Banach space $B^*(H)$ on the measurable space with the finite additive measure and the study of the dependence of this process properties on the choice of the measure on the regularizing parameters set.

The symmetric operator $\mathbf{L}$ can be extended to the self-adjoint operator in the wider extended Hilbert space. The regularization of Cauchy problem (1.1)–(1.3) can be considered as the unitary dynamics of pure state in the extended Hilbert space, namely, in the space of H-valued functions on the set of regularizing parameters which is square integrable with respect to some finite additive measure on its domain (see Ref. 9).

The mean value of above described stochastic process takes the two different roles. Firstly, it coincides with the conditional expectation (see Refs. 17 and 14) of the pure state in extended Hilbert space on subalgebra of operators independent of the regularizing parameter. Secondly, it is one of the continuous single-valued branch of set-valued map $\Phi(t)$.

3. The Well-Posedness of Cauchy Problem

The analysis of Cauchy problem with degenerated Hamiltonian in Ref. 27 shows that the solvability of Cauchy problem and the properties of its regularization are defined by the spectral characteristics of degenerated operator $\mathbf{L}$ such as deficiency index and defect subspaces.

For the formulation of the setting of Cauchy problem (1.1)–(1.3) we investigate the domain of maximal operator corresponding to the differential expression (1.3). The domain of maximal operator $\mathbf{L}$ is the linear manifold

$$D(\mathbf{L}) = \{u(x) \in H : \mathbf{L}u \in H\} \tag{3.1}$$

of the functions of the space H such that the operation of the multiplication and the derivation in the expression (1.3) is correct, and its result $\mathbf{L}u$ is the element of the space H.

This requirement for the poblem (II) implies that restrictions of any function $u \in D(\mathbf{L})$ on the segments Δ_j and on the set Δ satisfy the inclusions

$$u|_{\Delta_j} \in W_2^1(\Delta_j), \quad u|_{\Delta} \in W_2^2(\Delta). \tag{3.2}$$

Hence any function $u \in D(\mathbf{L})$ has the traces on the common boundary γ of the ellipticity domain with the domain of degeneration both from the side of ellipticity domain $u_e(\gamma)$ and from the side of degeneration domain $u_d(\gamma)$; moreover, the first derivative of this function has the trace $(\frac{\partial u}{\partial x}(\gamma))_e$ from the side of ellipticity domain. Then the inclusion (3.1) for the function satisfying condition (3.2) is equivalent to the condition of continuity of the function u

$$u_e(x) = u_d(x), \quad x \in \gamma, \tag{3.3}$$

and the condition of the continuity of the flow $j(x) = g(x)\frac{\partial u(x)}{\partial x} + \frac{i}{2}a(x)u(x)$:

$$j_e(x) = j_d(x), \quad x \in \gamma. \tag{3.4}$$

For example, in the case of problem (I) with the degeneration on the positive semiaxe the domain of operator $\mathbf{L}$ is the linear manifold

$$D(\mathbf{L}) = \left\{u|_{R_-} \in W_2^2(R_-), u|_{R_+} \in W_2^1(R_+), u(-0) = u(+0), u'(-0) = i\frac{b}{2}u(0)\right\}.$$

The operator $\mathbf{L}$ with the domain given by conditions (3.2)–(3.4) is densely defined, symmetric and closed since $\mathbf{L} = \mathbf{L}^{**}$. The linear manifold $D(\mathbf{L})$ supplied by the graph norm of operator $\mathbf{L}$ is Hilbert space which is also denoted by $D(\mathbf{L})$.

Definition 3.1. Function $u(t) \in C(R_+, D(\mathbf{L})) \bigcap C^1(R_+, H)$ is called the solution of Cauchy problem (1.1)–(1.3) if it satisfies Eq. (1.1) and initial condition (1.2).

Function $u(t) \in C(R_+, H)$ is called the generalized solution of Cauchy problem (1.1)–(1.3) if there is the sequence $\{u_{0k}\}$ of vectors of the space H such that the following requirements hold:

(1) Cauchy problem for Eq. (1.1) with the initial condition u_{0k} has the unique solution $u_k(t)$ for any $k \in \mathbb{N}$,
(2) $\lim\limits_{k\to\infty} \|u_{0k} - u_0\|_H = 0$,
(3) $\lim\limits_{k\to\infty} \|u_k(t) - u(t)\|_{C(R_+,H)} = 0$.

For the investigation of well-posedness and regularization of problem (1.1)–(1.3) we consider the conjugate operator $\mathbf{L}^*$ and the semigroups generated by operators $\pm i\mathbf{L}$ and $\pm i\mathbf{L}^*$. Operator $\mathbf{L}^*$ of problem (II) has wider domain

$$D(\mathbf{L}^*) = \left\{ u|_{R_-} \in W_2^2(R_-), u|_{R_+} \in W_2^1(R_+), \right.$$
$$\left. u'_e(x_j) + i\frac{b_{e,j}}{2}u_d(x_j) = ib_{d,j}u_d(x_j), x_j \in \gamma \right\},$$

where $b_{e,j}$ ($b_{d,j}$) are the limit values of function $b(x)$ in the boundary point $x_j \in \gamma$ at the side of the ellipticity (degeneration) domains of operator $\mathbf{L}$. Hence problem (1.1)–(1.3) is overdetermined.

Since $\mathrm{Re}(-i\mathbf{L}u, u) = 0$ for any $u \in D(\mathbf{L})$ and $\mathrm{Re}(i\mathbf{L}^*v, v) = \sum_{j=1}^n \frac{b_{d,j}}{2}|[v](x_j)|^2$ for any $v \in D(\mathbf{L}^*)$, where $[v](x_j) = v(x_j + 0) - v(x_j - 0)$, then (see Ref. 15) the operators $-i\mathbf{L}$ and $i\mathbf{L}$ are the conservative operators for any function $a(x)$ satisfying the considered requirements. If all values $b_{d,j}$ are nonpositive, then operator $-i\mathbf{L}$ is maximal dissipative operator in the space H, and if each of them is positive then the operator with opposite sign $i\mathbf{L}$ possesses the property of maximal dissipativity.

The deficient indices of the operator $\mathbf{L}$ can take nontrivial values (m_-, m_+) (here $m_\pm = \mathrm{dimKer}(\mathbf{L} \pm i\mathbf{I})$) in accordance with the signs of values $b_{d,j}$. If any value $b_{d,j}$ is nonzero then the relation $m_- + m_+ = n + 1$ holds. The following statement about the well-posedness of Cauchy problem is obtained in Ref. 27.

Theorem 3.1. *Cauchy problem* (1.1)–(1.3) *has a generalized solution for any* $u_0 \in H$ *if and only if the deficient indices of operator* $\mathbf{L}$ *are such that* $m_+ = 0$, m_- *is an arbitrary. If* $m_+ = 0$, *then operator* $\mathbf{L}$ *generates the isometric semigroup* $e^{-it\mathbf{L}} \equiv \mathbf{U}_{\mathbf{L}}(t), t > 0$, *in the space* H, *the solution* $u(t)$ *of problem* (1.1)–(1.3) *is unique, and it is obtained as the application of semigroup* $\mathbf{U}_{\mathbf{L}}(t), t > 0$, *to the vector*

u_0 *of the initial data*:

$$u(t) = \mathbf{U_L}(t)u_0, \quad t > 0.$$

If $m_+ \neq 0$, *then there is the infinitely-dimensional subspace* $H_1 \subset H$ *such that Cauchy problem* (1.1)–(1.3) *with an initial condition from the subspace* H_1 *has no generalized solution.*

Remark 3.1. In the case of problem (II) the condition $m_+ = 0$ is equivalent to the system of inequalities $b_{d,j} \leq 0$ for all $j \in \overline{1,n}$.

Corollary 3.1. *Let us consider operator* $\mathbf{L}$ *in problem* (I).

If $b \leq 0$, *then Cauchy problem* (1.1)–(1.3) *with an arbitrary initial data* $u_0 \in H$ *has the unique generalized solution* $u(t) = \mathbf{U_L}(t)u_0$.

If $b > 0$, *then* $H_1 = \overline{\bigcup_{t>0} \mathrm{Ker}((\mathbf{U_{-L}}(t))^*)}$. *In this case Cauchy problem* (1.1)–(1.3) *has a solution if and only if* $u_0 \in H_0 \equiv H \ominus H_1 = \overline{\bigcap_{t>0} \mathrm{Im}(\mathbf{U_{-L}}(t))}$. *If* $u_0 \in H_0$ *then the solution of Cauchy problem is unique and given by the formula* $u(t) = (\mathbf{U_{-L}}(t))^* u_0$.

Thus problem (1.1)–(1.3) is ill-posed if $m_+ \neq 0$. In particular, problem (I) is ill-posed if $b > 0$, and in this case Cauchy problem with initial data $u_0 \notin H_0$ has no solution.

4. Correct Linear Self-Adjoint Regularization of Cauchy Problem

Denote by $B(H_1, H_2)$ Banach space of linear bounded operators mapping some Hilbert space H_1 into another Hilbert space H_2 endowed with the operator norm.

Definition 4.1. The generalized sequence of Cauchy problems (1.2), (1.4) is called linear self-adjoint regularization of Cauchy problem (1.1)–(1.3) if it satisfies the following:

(1) linear operator $\mathbf{L}_\varepsilon$ is self-adjoint operator in the space H for any $\varepsilon \in E$;
(2) linear manifold $D = D(\mathbf{L}) \bigcap (\bigcap_{\varepsilon \in E} D(\mathbf{L}_\varepsilon))$ is dense in the space H,
(3) the equality $\lim_{\varepsilon \in E} \|\mathbf{L}_\varepsilon u - \mathbf{L}u\|_H = 0$ holds for any $u \in D$.

Linear self-adjoint regularization of Cauchy problem (1.1)–(1.3) is called correct regularization of order $q \in \mathbb{N}$ if the following holds

(4) for any $\varepsilon \in E$ there is linear bounded operator $\mathbf{Q}_\varepsilon$ acting from Hilbert space $D(\mathbf{L}^{q-1})$ into Hilbert space H which maps linear manifold $D(\mathbf{L}^q)$ into Hilbert space $D(\mathbf{L}_\varepsilon)$ and satisfies the conditions: $\sup_{\varepsilon \in E} \|\mathbf{Q}_\varepsilon\|_{B(D(\mathbf{L}^q), D(\mathbf{L}_\varepsilon))} < +\infty$, $\sup_{\varepsilon \in E} \|\mathbf{Q}_\varepsilon\|_{B(D(\mathbf{L}^{q-1}), H)} < +\infty$ and $\lim_{\varepsilon \in E}[\|\mathbf{Q}_\varepsilon u - u\|_H + \|\mathbf{L}_\varepsilon \mathbf{Q}_\varepsilon u - \mathbf{L}u\|_H] = 0$ for any $u \in D(L^q)$.

Let us note that if the sequence of linear operators $\{\mathbf{L}_n\}$ is correct regularization of Cauchy problem (1.1)–(1.3), then the graph of operator $\mathbf{L}$ (see Ref. 27) belongs to the limit set of the sequence of graphs $\{\Gamma_{\mathbf{L}_n}\}$ of regularizing operators in the topology of strong graph convergence.

We consider the following example of correct regularization of Cauchy problem (1.1)–(1.3):

Regularization (A): operators $\mathbf{L}_\varepsilon$ are given by the differential expression of type (3) with function $g_\varepsilon(x) = g(x) + \sum_{j=1}^{n} \varepsilon_j \chi_{\Delta_j}(x)$ instead of function $g(x)$, where $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$, $E = (0,1)^n, \varepsilon \to \varepsilon^* = (0, \dots, 0)$. Then (see Ref. 27) regularization (A) is correct regularization of order $q = 3$.

Let us fix some correct regularization of Cauchy problem (1.1)–(1.3) and consider the corresponding sequence of solutions of the regularizing problem $\{u_n(t)\}$. In Ref. 27 we investigate the properties of the solutions sequence of the regularizing problem such as its convergence and its compactness in the topologies of strong and weak convergence of the space H.

Theorem 4.1. *Let the sequence of self-adjoint operators* $\{\mathbf{L}_n\}$ *be the correct regularization of Cauchy problem* (1.1)–(1.3). *Then*

(1) *if the deficiency indices of operator* $\mathbf{L}$ *are* $(m_-, 0)$, *then operator* $-i\mathbf{L}$ *is the generator of isometric semigroup* $\mathbf{U_L}(t), t > 0$, *and the sequence of regularizing semigroups* $\{\mathbf{U}_{\mathbf{L}_n}(t)\}$ *converges to semigroup* $\mathbf{U_L}(t)$ *in the topology of strong operator convergence of the space* $B(H)$ *uniformly on an arbitrary segment* $[0, T], T > 0$;
(2) *if the deficiency indices of operator* $\mathbf{L}$ *are* $(0, m_+), m_+ \neq 0$, *then operator* $i\mathbf{L}$ *is the generator of isometric semigroup* $\mathbf{U}_{-\mathbf{L}}(t), t > 0$, *and the sequence of regularizing semigroups* $\mathbf{U_L}(t)$ *converges in the topology of weak operator convergence of the space* $B(H)$ *to the semigroup* $\mathbf{U}_{-\mathbf{L}}(t)^*$ *uniformly on an arbitrary segment* $[0, T], T > 0$.

Corollary 4.1. *If the deficiency indices of operator* $\mathbf{L}$ *are* $(m_-, 0)$, *then for any vector* $u_0 \in H$ *the sequence of regularizing solutions* $\{u_n(t)\}$ *converges to the generalized solution* $u(t)$ *of problem* (1.1)–(1.3) *in the space* H *uniformly on an arbitrary segment* $[0, T], T > 0$.

If the deficiency indices of operator $\mathbf{L}$ *are* $(0, m_+), m_+ \neq 0$, *then for any vector* $u_0 \in H$ *the sequence of regularizing solutions* $\{u_n(t)\}$ *converges to the function* $u^*(t) = \mathbf{U}_{-\mathbf{L}}(t)^* u_0$ *in the weak topology of the space* H *uniformly on an arbitrary segment* $[0, T], T > 0$.

Remark 4.1. If the operator $-i\mathbf{L}$ has maximal dissipative extension $-i\mathbf{A}$, then the stationary sequence of operators $\mathbf{L}_n = \mathbf{A}, n \in \mathbb{N}$, is the correct regularization of Cauchy problem (1.1)–(1.3) of order $q = 1$.

Theorem 4.2. *Let symmetric operator* $\mathbf{L}$ *be non-maximal dissipative operator. Then the validity of the inequality* $m_- \geq m_+$ *is the necessary condition of the existence of a subsequence of regularizing semigroups* $\{\mathbf{U}_{\mathbf{L}_k}(t)\}$ *which converges uniformly on an arbitrary segment in the strong operator topology of the space* $B(H)$.

If the condition $m_- \geq m_+$ *is satisfied and some subsequence of regularizing semigroups* $\{\mathbf{U}_{\mathbf{L}_k}(t)\}$ *converges in the strong operator topology of the space* $B(H)$ *to* *the operator-function $F(t), t > 0$ uniformly on any segment $[0, T]$ then the function $F(t), t > 0$, is isometric semigroup which is generated by one of maximal dissipative conservative extension of operator $-i\mathbf{L}$.*

The proof of Theorem 4.2 is given in Ref. 27.

Corollary 4.2. *If operator $-i\mathbf{L}$ has no maximal dissipative extension, then for any $t > 0$ an arbitrary subsequence of regularizing unitary operators $\mathbf{U}_{\mathbf{L}_{\varepsilon_n}}(t), \varepsilon_n \to \varepsilon^*$, diverges in the strong operator topology of the space $B(H)$.*

Thus the approximation approach exhibits the stability properties with respect to the choice of the correct regularization. The properties of the solutions sequence of regularizing problems such that convergence, compactness, the set of partial limits have no dependence on the choice of the correct regularization, and these properties are defined by the spectral properties (such as deficiency indexes and maximal dissipative extensions) of Hamiltonian of the original Cauchy problem.

5. Convergence of Correctly Regularizing Dynamical Semigroups

5.1. *The set of partial limits*

Let us note that the quantum state is the non-negative normalized linear functional on Banach space of bounded linear operators $B(H)$. The quantum state is called the normal state if it is continuous not only in the norm topology of $B(H)$ but also in the strong operator topology. The state can be presented as an element of Banach space $B^*(H)$, and the normal state can be presented as normalized trace class operator (density operator). The extreme points of the convex set of normal quantum states are called pure states. The trace class operator corresponding to the pure state is one-dimensional orthogonal projector in the space H. The necessity of consideration of non-normal states arises in the investigation of measurement of continuous observables (see Ref. 29).

The dynamical semigroup is called the semigroup of transformations of the set Σ of quantum states, i.e. the semigroup of maps $T_t, t > 0$, of the space B^* (see Refs. 5 and 14) which satisfies the following:

(1) $T_t \cdot T_s \rho(\mathbf{A}) = T_{t+s}\rho(\mathbf{A}), \forall\, \mathbf{A} \in B(H)$ for any $t, s \in R_+$, $\rho \in B^*(H)$;
(2) $T_0\rho = \rho$ for any $\rho \in B^*(H)$;
(3) the semigroup $\{T_t\}$ is continuous in weak* topology of the space $B^*(H)$, i.e. for any $\rho \in B^*(H)$, $\mathbf{A} \in B(H)$ the function $T_t\rho(\mathbf{A})$ is continuous on semiaxe R_+.

Any regularizing Cauchy problem generates the regularizing dynamical semigroup of maps of the set of quantum states $T_t(\varepsilon), t > 0, \varepsilon \in E$ acting on an arbitrary element $\rho \in B^*(H)$ by the rule

$$T_t(\varepsilon)\rho = \rho_\varepsilon(t),$$

where $\rho_\varepsilon(t)(\mathbf{A}) = \rho(\mathbf{U}^*_{\mathbf{L}_\varepsilon}(t)\mathbf{A}\mathbf{U}_{\mathbf{L}_\varepsilon}(t))$ for any $\mathbf{A} \in B(H)$.

Cauchy problem (1.1)–(1.3) defines for any initial condition $u_0 \in H$ the following generalized sequence of regularizing density operators

$$\rho_\varepsilon(t, u_0) = T_t(\varepsilon)\rho_{u_0}, \quad t > 0, \ \varepsilon \in E,$$

where $\rho_{u_0}(\mathbf{A}) = (u_0, \mathbf{A}u_0)$.

We fix some $t > 0$ and investigate the convergence of the generalized sequence of trajectories $\rho_\varepsilon(t, u_0), \varepsilon \in E$, of regularizing dynamical semigroups in the weak* topology of the space $B^*(H)$. For this aim we study the convergence of the numerical generalized sequences $\rho_{u_0}(\mathbf{U}^*_{\mathbf{L}_\varepsilon}(t)\mathbf{A}\mathbf{U}_{\mathbf{L}_\varepsilon}(t)), \varepsilon \in E$, with all possible $\mathbf{A} \in B(H)$.

Theorem 5.1. *If the sequence of solutions of regularizing problems* $\{u_{\varepsilon_n}(t)\}$ *converges for some* $t > 0$ *weakly in* H, *but not in the norm topology, then the corresponding sequence of density operators* $\rho_{\varepsilon_n}(t)$ *diverges in weak** *topology of the space* $B^*(H)$.

The statement of Theorem 5.1 followes from Theorem 1 of Ref. 8 (see also Ref. 26). According to Ref. 8 the convergence of the sequence of functionals $\rho_{\varepsilon_n}(t)$ in weak* topology of the space $B^*(H)$ is sufficient for the compactness of the sequence $\{u_{\varepsilon_n}(t)\}$ in the space H. Hence the following statement takes place for problem (I) (see Ref. 25): if $b > 0$ and $u_0 \notin H_0$ then there is the number $T^* \geq 0$ such that for any $t > T^*$ there exists the operator $\mathbf{A} \in B(H)$ with the divergent numerical sequence $\{\rho_{\varepsilon_n}(t)(\mathbf{A})\}$.

The convergence in the norm of $B(H)$ of subsequences of the sequence of regularizing density operators takes place under the assumption $m_- \geq m_+$. According to Theorem 4.2 the set of all possible partial limits of the sequence of regularizing density operators is the collection of density operators $\rho_\phi(t) = \mathbf{U}^*_{\mathbf{L}_\phi}(t)\rho_0\mathbf{U}\mathbf{L}_\phi(t)$, where $-i\mathbf{L}_\phi, \phi \in \Phi$, is the set of all possible maximal dissipative extensions of operator $-i\mathbf{L}$.

If $m_- < m_+$, then according to Theorem 5.1 and Corollary 4.2 the following statement holds. For any $t > 0$ there is some infinitely-dimensional subspace H_1 of the space H such that for any initial data $u_0 \in H_1$ the corresponding sequence of regularizing density operators diverges in weak* topology of the space $B^*(H)$.

Consider real Banach space $B_s(H)$ of bounded linear self-adjoint operators with the operator norm and the metric space $X = R \times H \times B_s(H)$ provided by the metric of the direct product of Banach spaces (see Ref. 7, Sec. 1.8). We investigate the generalized sequence of the functions $\{f_\varepsilon(t, u_0, \mathbf{A}), \varepsilon \in E\}$ defining on the space X by the equality $f_\varepsilon(t, u_0, \mathbf{A}) = (u_\varepsilon(t), \mathbf{A}u_\varepsilon(t)), \varepsilon \in E$.

The regularization (1.2), (1.4) of ill-posed Cauchy problem (1.1)–(1.3) generates (see Ref. 26) the set-valued map $F(\cdot) : X \to 2^R$ which acts by the rule

$$F(t, u_0, \mathbf{A}) = Ls_{\varepsilon\to 0} f_\varepsilon(t, u_0, \mathbf{A}), \tag{5.1}$$

where $Ls_{\varepsilon\to 0} f_\varepsilon(t, u_0, \mathbf{A})$ is the set of all partial limits of numerical sequence $\{f_\varepsilon(t, u_0, \mathbf{A}) = \rho_\varepsilon(t)(\mathbf{A})\}$. According to the results of Ref. 26, the following statement holds.

Theorem 5.2. *Let the generalized sequence of operators* $\{\mathbf{L}_\varepsilon, \varepsilon \in E\}$ *be the correct regularization of Cauchy problem* (1.1)–(1.3). *Then the map* F *is a continuous map of the metric space* X *into the metric space* 2^R *with Hausdorff metric. Moreover, if* E *is a connected set in some metric space and function* $u_\varepsilon(t)$ *is the continuous map of* E *into Hilbert space* H *for an arbitrary* $t > 0$, *then the set* $F(x)$ *is the segment for any point* $x \in X$.

Proof. The continuity of the set-valued function $F(x)$ on the metric space X with respect to Hausdorff metric is followed from the two estimates.

(1) Since the semigroups $\mathbf{U}_{\mathbf{L}_\varepsilon}(t), t > 0; \varepsilon \in E$, is unitary, the inequality

$$|f_\varepsilon(t, u_1, \mathbf{A}) - f_\varepsilon(t, u_2, \mathbf{A})| \leq 2\|\mathbf{A}\|_{B(H)}\|u_2 - u_1\|_H, \tag{5.2}$$

holds for any $\varepsilon \in E$ and any $u_1, u_2 \in H : \|u_1\| = \|u_2\| = 1$.

(2) If $u_1 \in D$ (see Definition 4.1, Condition 3), then the equality

$$\lim_E \|\mathbf{L}_\varepsilon u_1 - \mathbf{L}u_1\|_H = 0$$

follows from Definition 4.1. Hence the sequence $\{\|\mathbf{L}_\varepsilon u_1\|_H, \varepsilon \in E\}$ is bounded. If $(t_i, u_i, \mathbf{A}_i) \in X, i = 1, 2$, such that $u_1 \in D$, then the estimate

$$\begin{aligned} &|f_\varepsilon(t_1, u_1, \mathbf{A}_2) - f_\varepsilon(t_2, u_2, \mathbf{A}_2)| \\ &\quad \leq \|\mathbf{A}_2 - \mathbf{A}_1\|_{B(H)} + 2\|\mathbf{A}_1\|_{B(H)}\|u_2 - u_1\|_H + k\|\mathbf{A}_1\|_{B(H)}\|u_1\|_{D(\mathbf{L})}|t_2 - t_1| \end{aligned} \tag{5.3}$$

is valid for any $\varepsilon \in E$ with some $k > 0$ (see the property of the semigroups $\mathbf{U}_{\mathbf{L}_\varepsilon}(t), t > 0; \varepsilon \in E$ to be unitary).

If for given $x \in X$ the map $f_\varepsilon(x) : E \to R$ is continuous and the set E is connected, then image $F(x)$ is a connected set in R (see Ref. 26). □

Physical sense of the set-valued map F is the following. Let us assume that the quantum system is subjected to dynamics defining by Eq. (1.1) with regularization (1.4). Then an arbitrary number of the set $F(t, u_0, \mathbf{A})$ can be the mean value of the observable $\mathbf{A} \in B(H)$ at the time t for the quantum system with the initial vector u_0 of the quantum state.

5.2. *Measures on the set of subsequences*

For the investigation of the probabilistic properties of the set-valued map (5.1) we consider the set $W(E)$ of non-negative normalized measures μ defining on the algebra 2^E of all subsets of the directed set E and satisfying:

if $A \in 2^E$, *and there is the element* $e \in E$ *such that* $t \prec e$ *for all* $t \in A$, *then* $\mu(A) = 0$ (see Refs. 12 and 30).

In Ref. 26 using Hahn–Banach theorem, we proved that the set $W(E)$ is nonempty and convex (see also Refs. 7 and 3). For example, if $E = \mathbb{N}$, then $W(E)$

consists of the part of the unit sphere of the space $l_\infty^* \backslash l_1$ lying in the non-negative cone. Let us note that any measure from the set $W(E)$ is finitely additive and is singular with respect to an arbitrary countable additive measure on the set E.

We choose in the set of measures $W(E)$ the special class $W_0(E)$ of measures possessing the following special properties: any measure of class $W_0(E)$ takes only two values: 0 and 1. The existence of measures with these properties can be proved with the use of the transfinite induction method (see Ref. 12).

Remark 5.1. The set $W_0(E)$ consists of the extreme points of the convex set $W(E)$. The proof of this statement can be given analogously as the proof of Theorem 33 of Ref. 30.

5.3. *The space of functions integrable with respect to finitely additive measures*

Let us assume that the non-negative normalized finitely additive measure μ is defined on the measurable space $(E, 2^E)$. Then any bounded real-valued function $f(\varepsilon), \varepsilon \in E$ is measurable with respect to measure μ, and the description of Radon integral of this function (with respect to μ) is given in the monograph, Ref. 13.

Let us consider some Banach space Y. The procedure of integration of real-valued and Y-valued function on the measurable space with finitely additive measure is described in Refs. 7 and 9. In the same place the space $L_1(E, 2^E, \mu, Y)$ is defined as the supplement of the space of simple Y-valued functions on the set E (which takes only finite number of values in the space Y). In Ref. 28 we introduce the space $\mathcal{L}_1(E, 2^E, \mu, Y)$ as the supplement of linear space $b(E, Y)$ of bounded Y-valued functions on the set E endowed with the norm $\|f\|_{L_1(E,2^E,\mu,Y)} = \int_E \|f(\varepsilon)\|_Y d\mu$. In Ref. 28 we show that Banach space $\mathcal{L}_1(E, 2^E, \mu, Y)$ includes the space $L_1(E, 2^E, \mu, Y)$. If the space Y is finite-dimensional or measure μ is countable additive, then the space $L_1(E, 2^E, \mu, Y)$ coincides with the space $\mathcal{L}_1(E, 2^E, \mu, Y)$.

Let us consider the space $\mathcal{L}_2(E, 2^E, \mu, H) \equiv \mathcal{H}$ which is the supplement of the linear space of bounded H-valued functions on the set E endowed with the norm $\|f\|_{L_2(E,2^E,\mu,H)} = (\int_E \|f(\varepsilon)\|_H^2 d\mu)^{\frac{1}{2}}$. Then the space $\mathcal{H}$ is Hilbert space which includes the space $L_2(E, 2^E, \mu, H)$ (see Ref. 7) as the closed subspace.

In Ref. 10 the weak integral by Pettis and Gel'fand is defined for Y-valued functions.

Definition 5.1. Let a function $f(\varepsilon), \varepsilon \in E$, be defined on the measurable space $(E, 2^E, \mu)$ and takes values in the space Y. The element $z \in Y$ is called the weak* integral of $f \in \mathcal{L}_1(E, 2^E, \mu, Y)$ with respect to measure μ if for any element $y \in Y_*$ the following relation holds

$$z(y) = \int_E y(f(\varepsilon)) d\mu(\varepsilon).$$

(Here Y_* is the predual space of Banach space Y.)

5.4. *Continuous single-valued branches of set-valued maps*

Now we apply the procedure of the weak* integration by measure $\mu \in W(E)$ to the generalized sequence $\{\rho_\varepsilon(t, u_0)\}$.

Remark 5.2. Let us consider the regularization (A) of problems (I) and (II) as the maps $\varepsilon \to u_\varepsilon(t)$ of the domain E of finite-dimensional Euclidean space into Banach space $C(R_+, H)$. Then according to the results of Refs. 27 and 26 the map $u_\varepsilon(t) : E \to C(R_+, H)$ is continuous in any point of the set E.

Theorem 5.3. *Let E be an arcwise connected set in some metric space and the map $u_\varepsilon(t) : E \to C(R_+, H)$ be continuous. Then for any $u_0 \in H$ and any $\mu \in W(E)$ the following statements hold:*

(1) *The function $\rho^*_\mu(t, u_0) = \int_E \rho_\varepsilon(t, u_0)d\mu$ is the continuous map in the topology of weak* convergence of metric space $R_+ \times H$ into Banach space $B^*(H)$.*
(2) *The function $f^*_\mu(t, u_0, \mathbf{A}) = \rho^*_\mu(t, u_0)(\mathbf{A})$ is the continuous single-valued branch of the set-valued map $F(t, u_0, \mathbf{A})$.*
(3) *There is the set-valued map $\Phi(t, u_0)$ which is defined on the metric space $R_+ \times H$ and acts into the metric space $2^{B^*(H)}$ such that for any $\mathbf{A} \in B(H)$ the following equality holds $\Phi(t, u_0)(\mathbf{A}) = F(t, u_0, \mathbf{A})$.*

Proof. Let us assume that $u_0 \in H$. The set of all particular limits $F(t, u_0, \mathbf{A})$ is the segment according to Theorem 5.2. Hence for any $\mu \in W$ the following inclusion holds $f^*_\mu(x) \in F(x)$, where $f^*_\mu(x) = \int_E f_\varepsilon(x)d\mu$ and $f_\varepsilon(t, u_0, \mathbf{A}) = \rho_\varepsilon(t, u_0)(\mathbf{A})$. Therefore function $f^*_\mu(x)$ is the single-valued branch of set-valued map (5.1). The continuity of function $f^*_\mu(x)$ on the metric space X follows from (5.2) and (5.3).

For given $(t, u_0) \in R \times H$ we define the element $\rho^*_\mu(t, u_0) \in B^*(H)$ as the weak* integral of the function $\rho_\varepsilon(t, u_0), \varepsilon \in E$, by the measure μ. Then the function $\rho^*_\mu(t, u_0)$ is defined on the set $R \times H$, takes values in the space $B^*(H)$, satisfies the equality $\rho^*_\mu(t, u_0)(\mathbf{A}) = f^*_\mu(t, u_0, \mathbf{A})$ and estimates $|\rho^*_\mu(t, u_0)(\mathbf{A})| \le \|\mathbf{A}\|_{B(H)}$ for any element $\mathbf{A} \in B(H)$. The weak* continuity of the function $\rho^*_\mu(t, u_0)$ as the map of metric space $R \times H$ into Banach space $B^*(H)$ follows from the continuity of the real-valued function $f^*_\mu(t, u_0, \mathbf{A})$ on the space X.

Any point P of the set $F(x)$ is the limit of some subsequence $f_{\varepsilon_n}(x)$. If the measure $\mu \in W(E)$ is concentrated on the set K of values of the sequence ε_n, then $P = f^*_\mu(x)$. Since the directed set K is the subset of the set E then the set of measures $W(E)$ contains the subclass of measures $W(K)$ whose supports belong to the set K. Therefore $P \in \bigcup_{\mu \in W(E)} f^*_\mu(x)$, and the inclusion $F(x) \subset \bigcup_{\mu \in W(E)} f^*_\mu(x)$ holds for any $x \in X$. Since the function $f^*_\mu(x)$ is the continuous branch of the map $F(x)$, then the equality $F(x) = \bigcup_{\mu \in W(E)} f^*_\mu(x)$ holds for any $x \in X$.

Let us define the set-valued map $\Phi(t, u_0) : R \times H \to 2^{B^*(H)}$ acting by the rule

$$\Phi(t, u_0) = \bigcup_{\mu \in W(E)} \rho^*_\mu(t, u_0).$$

Then for any $\mu \in W(E)$ the map ρ^*_μ is the weak* continuous single-valued branch of the set-valued map Φ. Moreover, for any $(t, u_0, \mathbf{A}) \in X$ the equality $\Phi(t, u_0)(\mathbf{A}) = F(t, u_0, \mathbf{A})$ holds. □

Remark 5.3. If some sequence of solutions of the regularizing problem $\{u_{\varepsilon_k}\}$ converges weakly but not strongly in the space H, and measure μ belongs to class $W(\mathbb{N})$, then functional $\rho^*_\mu(t, u_0) = \int_{\mathbb{N}} \rho_{\varepsilon_k}(t, u_0) d\mu \in B^*(H)$ is not the trace class operator (normal state) (see Ref. 26).

Remark 5.4. An element $\rho \in B^*(H)$ belongs to the set $\Phi(t, u_0)$ if and only if for any operator $\mathbf{A} \in B(H)$ there is the subsequence $\{\varepsilon_k\}$ of regularizing parameters such that $\lim_{k\to\infty} \rho_{\varepsilon_k}(t, u_0)(\mathbf{A}) = \rho(\mathbf{A})$.

5.5. *Regularization as the dynamics in extended space*

Regularization (1.2), (1.4) of Cauchy problem (1.1), (1.2) with measure μ on the algebra of all subsets of the set of regularization parameters defines in the space $\mathcal{H}$ the unitary group $\mathcal{U}(t), t \in R$, of the transformation which acts by the rule

$$\mathcal{U}(t)u(x, \varepsilon) = \mathbf{U}_{\mathbf{L}_\varepsilon}(t)u(x, \varepsilon).$$

In fact, if arbitrary two elements $u(x, \varepsilon), v(x, \varepsilon)$ belong to linear manifold $b(E, H)$ of bounded H-valued functions on the set E (dense in the space $\mathcal{H}$), then the following equation $(\mathcal{U}(t)u(x, \varepsilon), \mathcal{U}(t)v(x, \varepsilon))_{\mathcal{H}} = (\mathbf{U}_{\mathbf{L}_\varepsilon}(t)u(x, \varepsilon), \mathbf{U}_{\mathbf{L}_\varepsilon}(t)v(x, \varepsilon))_{\mathcal{H}} = \int_E (\mathbf{U}_{\mathbf{L}_\varepsilon}(t)u(x, \varepsilon), \mathbf{U}_{\mathbf{L}_\varepsilon}(t)v(x, \varepsilon))_H d\mu = (u(x, \varepsilon), v(x, \varepsilon))_{\mathcal{H}}$ holds for any $t \in R$. Therefore the transformation $\mathcal{U}(t)$ of the space $\mathcal{H}$ is unitary for an arbitrary $t \in R$, since an arbitrary vector $u(x, \varepsilon) \in b(E, H)$ has the preimage $\mathbf{U}_{\mathbf{L}_\varepsilon}(-t)u(x, \varepsilon) \in b(E, H)$.

The generator of semigroup $\mathcal{U}(t), t \in R$, is self-adjoint operator $\mathcal{L}$ in the space $\mathcal{H}$ such that its domain $D(\mathcal{L})$ includes any function $u(x, \varepsilon)$ which satisfies the condition $u(x, \varepsilon) \in D(\mathbf{L}_\varepsilon)$ in μ-a.e. $\varepsilon \in E$, and for any function $u(x, \varepsilon) \in D(\mathcal{L})$ the equation $\mathcal{L}u(x, \varepsilon) = \mathbf{L}_\varepsilon u(x, \varepsilon)$ holds for μ-a.e. $\varepsilon \in E$.

A unit vector $u = u(x, \varepsilon) \in \mathcal{H}$ defines the pure quantum state in the space $\mathcal{H}$: the density operator $\mathcal{R}_u \in B^*(\mathcal{H})$ acting on an arbitrary operator $\mathbf{A} \in B(\mathcal{H})$ by the rule $\mathcal{R}_u(\mathbf{A}) = (u, \mathbf{A}u)_{\mathcal{H}}$. Thus regularization (1.2), (1.4) of Cauchy problem (1.1), (1.2) defines the unitary dynamics of vectors of the space $\mathcal{H}$ and the dynamical semigroup of transformations of density operators $\mathcal{R} \to \mathcal{R}(t) = \mathcal{U}(t)\mathcal{R}\mathcal{U}^*(t)$.

Remark 5.5. According to Naymark theorem the symmetric operator $\mathbf{L}$ in the space H can be extended to a self-adjoint operator $\mathcal{L}$ in some Hilbert space $\mathcal{H}$ including the space H as the subspace. In this work we construct the concrete realization of one of the extensions.

5.6. *Conditional expectation of the state on the extended space*

The restriction of functional f on subalgebra B_1 of algebra $B(H)$ is called the partial trace or the conditional expectation of some state $f \in B^*(H)$ on subalgebra B_1 (see Refs. 14 and 9).

Let us denote by $B_0(\mathcal{H})$ the linear subspace of the space $B(\mathcal{H})$ which consists of the operators $\mathbf{A}$ satisfying the following condition:

there is the operator $\mathbf{A}_0 \in B(H)$ *such that the image* $\mathbf{A}u(\varepsilon)$ *of any* H*-valued bounded function* $u(\varepsilon) \in b(E,H)$ *is the function of the space* $b(E,H)$ *defining by the equality* $(\mathbf{A}u)(\varepsilon) = \mathbf{A}_0(u(\varepsilon))$ *pointwisely on the set* E.

Thus the bounded densely defined operator $\mathbf{A}$ has the unique continuous continuation on the space $\mathcal{H}$. We say that any operator $\mathbf{A}$ of the subspace $B_0(\mathcal{H})$ is the operator independent on the variable ε. We denote such operator by the symbol $\mathbf{A} = \mathbf{A}_0 \times \mathbf{I}_1$. Linear subspace $B_0(\mathcal{H})$ is also the subalgebra of algebra $B(\mathcal{H})$ according to the rule of the action of this operator on the vectors of dense linear manifold $b(E,H)$.

Remark 5.6. Since measure μ is the finite additive, then Hilbert space $\mathcal{L}_2(E, 2^E, \mu, H)$ is nonseparable and is not representable as the tensor product of the spaces $\mathcal{L}_2(E, 2^E, \mu, \mathbf{C})$ and H (see Ref. 28), if $\mu \in W_0(E)$.

Definition 5.2. The restriction of functional $f \in B^*(\mathcal{H})$ on linear subspace $B_0(\mathcal{H})$ is called the partial trace of functional $f \in B^*(\mathcal{H})$ with respect to linear space $\mathcal{L}_2(E, 2^E, \mu, \mathbf{C})$.

The partial trace is the continuous map of space $B^*(\mathcal{H})$ into Banach space $B_0^*(\mathcal{H})$ (see Ref. 4). The following claim is the corollary of Theorem 5.3 and Definition 5.2.

Corollary 5.1. *Quantum state* $\rho_\mu^*(t, u_0)$ *is the partial trace of pure state* $\mathcal{R}(t) \in B^*(\mathcal{H})$ *with respect to linear space* $\mathcal{L}_2(E, 2^E, \mu, \mathbf{C})$.

5.7. *The regularization as the stochastic process*

The aim of this part of the paper is to consider the diverging sequence of density operators $\rho_\varepsilon(t)$ as the stochastic process on the measurable space $(E, 2^E)$ with some measure $\mu \in W(E)$ taking values in Banach space $B^*(H)$ and defining by the regularization of Cauchy problem (1.1)–(1.3).

But, in order to interpret the regularization in terms of stochastic processes, we should use the term "stochastic process" in the case when "probability measure" μ is finitely additive but not countably additive, in general. Therefore we should generalize the notion of stochastic process.

Definition 5.3. Let the measurable space $(E, 2^E)$ be endowed by the (finite additive) non-negative normalized measure μ defining on σ-algebra 2^E. Let the Banach space Y endowed by the structure of measurable space with some σ-algebra of subsets $\mathcal{B}(Y)$. We say that the function f mapping the set $R_+ \times E$ into some Banach space Y is called *Y-valued stochastic process* (*on the space with measure* $(E, 2^E, \mu)$) if for any $t > 0$ the map $f(t,\cdot) : E \to Y$ is measurable with respect to pair of σ-algebras on measurable spaces $(E, 2^E)$ and $(Y, \mathcal{B}(Y))$. The *mean value* of Y-valued stochastic process f is the function $\bar{f} : R_+ \to Y$ where $\bar{f}(t) = \int_E f(t,\varepsilon)d\mu$ (see Definition 5.1).

For given $u_0 \in H$ we consider the process with trajectories $\rho_\varepsilon(t), \varepsilon \in E$, where $(\rho_\varepsilon(t), \mathbf{A}) = (u_\varepsilon(t), \mathbf{A}u_\varepsilon(t)), \mathbf{A} \in B(H)$, and $u_\varepsilon(t) = \mathbf{U}_{\mathbf{L}_\varepsilon}(t)u_0$.

Let us fix some $u_0 \in H$ and some $t > 0$. Then for any operator $\mathbf{A} \in B(H)$ measure μ induces measure $m_\mu^{\mathbf{A}}$ on the real line R by the formula

$$m_\mu^{t,u_0,\mathbf{A}}(\Delta) = \sup\{\mu(K) : K \in 2^E, F_K(t, u_0, \mathbf{A}) \subset \Delta\},$$

where $F_K(t, u_0, \mathbf{A})$ is the set of all partial limits of the generalized numerical sequence $f_\varepsilon(t, u_0, \mathbf{A}),\ \varepsilon \in E \bigcap K, \varepsilon \to \varepsilon^*$ (see Ref. 26).

For given finite collection of the operators $\alpha = \mathbf{A}_1, \dots, \mathbf{A}_N \in B(H)$ and the intervals $\tau = \Delta_1, \dots, \Delta_N \subset R$ we denote by $S_{\alpha,\tau}$ the cylindrical set $S_{\alpha,\tau} = \{f \in B^*(H) : f(\mathbf{A}_j) \in \Delta_j, j \in 1, \dots, N\}$ in the space $B^*(H)$. The union of the cylindrical sets $S_{\alpha,\tau}$ forms the semiring of subsets of the space $B^*(H)$. Therefore according to Kolmogorov theorem on the measure continuation the following statement holds: the measure μ induces the non-negative normalized finitely additive measure M_{t,u_0}^μ on the algebra of cylindrical sets $\mathcal{S}(B^*(H))$ such that

$$M_{t,u_0}^\mu(S_{\mathbf{A},\Delta}) = m_\mu^{t,u_0,\mathbf{A}}(\Delta) = \sup\{\mu(K) : K \in 2^E, F_K(t, u_0, \mathbf{A}) \subset \Delta\}$$

for any operator $\mathbf{A}$ and any interval Δ. Here the collection $\mathcal{S}(B^*(H))$ of subsets of the space $B^*(H)$ is the minimal ring over the semiring of subsets of type $S_{\alpha,\tau}$. Thus if the set of regularization parameters E is endowed by measure μ from the class $W(E)$, then the regularization (1.2), (1.4) of Cauchy problem (1.1)–(1.3) defines the stochastic process $\rho_\varepsilon(t, u_0)$ on the space with measure $(E, 2^E, \mu)$ taking values in the measurable space $(B^*(H), \mathcal{S}(B^*(H))$. Our definition of stochastic process is the extension of definition of cylindrical stochastic process in Ref. 6: the measure M_{t,u_0}^μ on the algebra of cylindrical sets in the space $B^*(H)$ can only be finite additive.

The following property of the stochastic process $\rho_\varepsilon(t, u_0)$ is the corollary of Theorem 5.3:

Corollary 5.2. *Let $\mu \in W(E)$ and $\{L_\varepsilon, \varepsilon \in E\}$ be some correct regularization of Cauchy problem. Then function $\rho_\mu^*(t, u_0)$ is the mean value of the stochastic process $\rho_\varepsilon(t, u_0)$ on the space with measure $(E, 2^E, \mu)$.*

Moreover, the regularization of Cauchy problem (1.1)–(1.3) with measure $\mu \in W(E)$ on the set of regularization parameters defines the one-parameter family $\mathcal{T}_t, t > 0$, of continuous transformations of the space of measures on algebra

$\mathcal{S}(B^*(H))$ by the rule:

$$\mathcal{T}_t(M_0)(S_{\mathbf{A},\Delta}) = \int_E M_0(S_{\mathbf{U}_{\mathbf{L}_\varepsilon}(t)^*\mathbf{A}\mathbf{U}_{\mathbf{L}_\varepsilon}(t),\Delta})d\mu, \quad t > 0,$$

for any non-negative normalized measure M_0 on algebra $\mathcal{S}(B^*(H))$, any operator $\mathbf{A}$ and interval Δ.

Hence, $\mathcal{T}_{+0}(M_0) = M_0$ for any measures M_0. However, the one-parameter family of continuous transformation $\mathcal{T}_t$ of the space of measures on algebra $\mathcal{S}(B^*(H))$ of cylindrical sets of the space $B^*(H)$ does not possess the semigroup property.

Similar qualitative properties are inherent in the transformations of the set of quantum states generating by the procedures of observable measurements (see Refs. 23 and 29). Both of the map $\mathcal{T}_t(\cdot)$ and the procedure of the observable measurement transform any pure state to the measure on the set of pure states.

Let us note that Ref. 22 is devoted to the study of the dynamics of measures generating by the sequences of solutions of the regularizing problem. The probabilistic approach to the investigation of solutions of incorrect boundary value problems for nonlinear partial differential equations (for example, Vlasov–Poisson system) is evolved in Ref. 2.

The mean value of stochastic processes $\rho_\varepsilon(t, u_0)$ on the measurable space $(E, 2^E, \mu)$ with measure μ from class $W(E)$ defines the dynamical map (see Ref. 14) $T_t^\mu, t > 0$, of the set Σ of quantum states into itself:

$$T_t^\mu \rho(\mathbf{A}) = \int_E \rho(\mathbf{A}_\varepsilon(t))d\mu(\varepsilon) \quad \forall\, \mathbf{A} \in B(H),$$

where $\mathbf{A}_\varepsilon(t) = \mathbf{U}_{\mathbf{L}_\varepsilon}(t)\mathbf{A}\mathbf{U}_{\mathbf{L}_\varepsilon}(t)^*$.

In general, the dynamical map $T_t^\mu, t > 0$, of the set of quantum states does not possess the semigroup properties:

$$T_t^\mu(T_s^\mu\rho)(\mathbf{A}) = T_{t+s}^\mu\rho(\mathbf{A}) \quad \forall\, \mathbf{A} \in B(H), \;\; \rho \in B^*(H).$$

This phenomenon is typical property for the transformation appearing in the procedure of the averaging of the collection of semigroup dynamics by using the partial trace operation (see Ref. 1).

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