Available online at www.sciencedirect.com
ScienceDirect
Linear Algebra and its Applications 422 (2007) 274–278
LINEAR ALGEBRA AND ITS APPLICATIONS
www.elsevier.com/locate/laa

Characterization of the trace by monotonicity inequalities ☆

Airat M. Bikchentaev, Oleg E. Tikhonov *

Research Institute of Mathematics and Mechanics, Kazan State University, Universitetskaya 17, Kazan 420008, Russian Federation

Received 10 March 2006; accepted 5 October 2006
Available online 22 November 2006
Submitted by R. Bhatia

Abstract

Let φ be a positive linear functional on the algebra of $n \times n$ complex matrices and p be a number greater than 1. The main result of the paper says that if for any pair A, B of positive semi-definite $n \times n$ matrices with $A \leqslant B$ the inequality $\varphi(A^p) \leqslant \varphi(B^p)$ holds true, then φ is a nonnegative scalar multiple of the trace.

Keywords: Characterization of the trace; Monotonicity

Throughout the paper, $\mathscr{M}_n$ stands for the algebra of $n \times n$ complex matrices, $\mathscr{M}_n^{\mathrm{h}}$ and $\mathscr{M}_n^+$ denote the subsets of Hermitian and positive semi-definite matrices, respectively. For $A, B \in \mathscr{M}_n^{\mathrm{h}}$, the notation $A \leqslant B$ means that $B - A \in \mathscr{M}_n^+$. A linear functional φ on $\mathscr{M}_n$ is said to be *positive* if $\varphi(A) \geqslant 0$ for all $A \in \mathscr{M}_n^+$. For a real-valued function f of a real variable and a matrix $A \in \mathscr{M}_n^{\mathrm{h}}$, the value $f(A)$ is understood by means of the functional calculus for Hermitian matrices.

The Löwner–Heinz inequality says that if $0 \leqslant p \leqslant 1$ then for any pair $A, B \in \mathscr{M}_n^+$ such that $A \leqslant B$, it holds $A^p \leqslant B^p$. It is well known that a weaker inequality $\mathrm{Tr}(A^p) \leqslant \mathrm{Tr}(B^p)$ still holds for every $p > 1$. The aim of the present paper is to show that the latter property can serve to characterize the trace among the positive linear functionals on $\mathscr{M}_n$.

☆ Supported by the Russian Foundation for Basic Research (grant no. 05-01-00799).
* Corresponding author.
E-mail addresses: Airat.Bikchentaev@ksu.ru (A.M. Bikchentaev), Oleg.Tikhonov@ksu.ru (O.E. Tikhonov).

0024-3795/$ - see front matter © 2006 Elsevier Inc. All rights reserved.
doi:10.1016/j.laa.2006.10.005

Lemma. *Let a function* $f\colon [0, +\infty) \to \mathbb{R}$ *satisfy the conditions*

(a) $f(0) = 0$,
(b) $f(x) = \mathrm{o}(x)$ $(x \to +0)$,
(c) $f(x)$ *is differentiable at the point* $x = 2$ *and* $f'(2) < f(2)$.

Let $S = \mathrm{diag}(\alpha, 1)$, *where* $0 \leqslant \alpha \leqslant 1$. *If for any pair* $A, B \in \mathscr{M}_2$ *such that* $0 \leqslant A \leqslant B$ *it holds*

$$\mathrm{Tr}(Sf(A)) \leqslant \mathrm{Tr}(Sf(B)), \tag{1}$$

then $\alpha = 1$.

Proof. First, we take $c > 0$ such that $1 - c\sqrt{\alpha} \geqslant 0$, and consider the matrices

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1+c & 1-c\sqrt{\alpha} \\ 1-c\sqrt{\alpha} & 1+c\alpha \end{pmatrix}.$$

Clearly, $0 \leqslant A \leqslant B$.

Let us present an explicit form of the spectral representation $B = \lambda_1 P_1 + \lambda_2 P_2$ (λ_1, λ_2 being eigenvalues of B, and P_1, P_2 being corresponding projection matrices):

$$\lambda_1 = 1 + c\left(\frac{1+\alpha}{2}\right) + \frac{1}{2}\sqrt{4 - 8c\sqrt{\alpha} + c^2(1+\alpha)^2},$$
$$\lambda_2 = 1 + c\left(\frac{1+\alpha}{2}\right) - \frac{1}{2}\sqrt{4 - 8c\sqrt{\alpha} + c^2(1+\alpha)^2},$$
$$P_1 = \begin{pmatrix} t & \sqrt{t(1-t)} \\ \sqrt{t(1-t)} & 1-t \end{pmatrix}, \quad P_2 = \begin{pmatrix} 1-t & -\sqrt{t(1-t)} \\ -\sqrt{t(1-t)} & t \end{pmatrix},$$

where

$$t = \frac{1}{2}\left(\frac{c(1-\alpha)}{\sqrt{4 - 8c\sqrt{\alpha} + c^2(1+\alpha)^2}} + 1\right).$$

Then, let us estimate

$$\begin{aligned} \mathrm{Tr}(Sf(B)) &= f(\lambda_1)\mathrm{Tr}(SP_1) + f(\lambda_2)\mathrm{Tr}(SP_2) \\ &= f(\lambda_1)((\alpha-1)t + 1) + f(\lambda_2)((1-\alpha)t + \alpha) \end{aligned} \tag{2}$$

with making use of the fact that if a function $g(c)$ is differentiable at the point $c = 0$ then it can be represented in the form $g(c) = g(0) + g'(0)c + \mathrm{o}(c)$ $(c \to 0)$. We have

$$\lambda_2 = 1 + c\left(\frac{1+\alpha}{2}\right) - \frac{1}{2}\big(2 - 2\sqrt{\alpha}c + \mathrm{o}(c)\big) = \frac{1}{2}\big(1 + \alpha + 2\sqrt{\alpha}\big)c + \mathrm{o}(c) \quad (c \to 0).$$

Since $f(x) = \mathrm{o}(x)$ $(x \to +0)$, it follows

$$f(\lambda_2) = \mathrm{o}(c) \quad (c \to +0). \tag{3}$$

Also, we have

$$f(\lambda_1) = f(2) + f'(2)\frac{1 + \alpha - 2\sqrt{\alpha}}{2}c + \mathrm{o}(c) \quad (c \to 0). \tag{4}$$

Since

$$t = \frac{1}{2} + \left(\frac{1-\alpha}{4}\right) c + \mathrm{o}(c) \quad (c \to 0), \tag{5}$$

we obtain, substituting (3)–(5) into (2),

$$\mathrm{Tr}(Sf(B)) = \frac{f(2)}{2}(\alpha+1) + \frac{1}{4}\left(f'(2)\left(1-\sqrt{\alpha}\right)^2(\alpha+1) - f(2)(1-\alpha)^2\right) c$$
$$+ \mathrm{o}(c) \quad (c \to +0).$$

Clearly,

$$\mathrm{Tr}(Sf(A)) = \frac{f(2)}{2}(\alpha+1).$$

Thus, we obtain as an implication of (1)

$$\begin{aligned} 0 &\leqslant \frac{1}{4}\left(f'(2)\left(1-\sqrt{\alpha}\right)^2(\alpha+1) - f(2)(1-\alpha)^2\right) c + \mathrm{o}(c) \\ &= -\frac{1}{4}\left(1-\sqrt{\alpha}\right)^2\left((f(2)-f'(2))(\alpha+1) + 2f(2)\sqrt{\alpha}\right)c + \mathrm{o}(c) \quad (c \to +0). \end{aligned} \tag{6}$$

Observe, that inequality (1) in the hypothesis of Lemma implies that f is non-decreasing, hence $f(2) \geqslant 0$, and we conclude that the inequality (6) can hold for all $c > 0$ with $1 - c\sqrt{\alpha} \geqslant 0$ only if $\alpha = 1$. $\square$

Theorem. *Let $1 < p < \infty$ and φ be a positive linear functional on $\mathscr{M}_n$, such that*

$$\varphi(A^p) \leqslant \varphi(B^p) \tag{7}$$

whenever $0 \leqslant A \leqslant B$. Then φ is a nonnegative scalar multiple of the trace.

Proof. First, we consider the case $1 < p < 2$. Observe that the function $f(x) = x^p (x \in [0, +\infty))$ satisfies the conditions (a)–(c) of Lemma. As is well known, every positive linear functional φ on $\mathscr{M}_n$ can be represented in the form $\varphi(\cdot) = \mathrm{Tr}(S\cdot)$ for some $S \in \mathscr{M}_n^+$. It is easily seen that without loss of generality we can assume that $S = \mathrm{diag}(\alpha_1, \alpha_2, \ldots, \alpha_n)$, and we have to prove that $\alpha_i = \alpha_j$ for all $i, j = \overline{1, n}$. Clearly, it suffices to prove that $\alpha_1 = \alpha_2$. Inequality (7) must hold, in particular, for all matrices $A = [a_{ij}]_{i,j=1}^n$, $B = [b_{ij}]_{i,j=1}^n$ in $\mathscr{M}_n^+$ such that $A \leqslant B$ and $0 = a_{ij} = b_{ij}$ if $3 \leqslant i \leqslant n$ or $3 \leqslant j \leqslant n$. Thus the proof of theorem in the case $1 < p < 2$ reduces to Lemma.

Now, consider the case $p \geqslant 2$. Take $q > 0$ such that $1 < pq < 2$ and set $r = pq$. Let $A, B \in \mathscr{M}_n$ be such that $0 \leqslant A \leqslant B$. Since $q < 1$, the Löwner–Heinz inequality gives $A^q \leqslant B^q$. Then by the hypothesis of Theorem, $\varphi((A^q)^p) \leqslant \varphi((B^q)^p)$, i. e. $\varphi(A^r) \leqslant \varphi(B^r)$. From the first part of the proof it follows that φ in a nonnegative scalar multiple of the trace. $\square$

Corollary 1. *Let φ be a positive linear functional on $\mathscr{M}_n$, such that for any pair $A, B \in \mathscr{M}_n^{\mathrm{h}}$ with $A \leqslant B$ the inequality*

$$\varphi\left(\mathrm{e}^A\right) \leqslant \varphi\left(\mathrm{e}^B\right) \tag{8}$$

holds. Then φ is a nonnegative scalar multiple of the trace.

Proof. Let $C, D \in \mathscr{M}_n^+$ be such that $C \leqslant D$. Since the function $\ln x$ is matrix monotone, $2\ln(C + t) \leqslant 2\ln(D + t)$ for every positive number t. Then

$$\varphi\big((C+t)^2\big) = \varphi\big(e^{2\ln(C+t)}\big) \leqslant \varphi\big(e^{2\ln(D+t)}\big) = \varphi\big((D+t)^2\big)$$

by (8), and the continuity arguments show that $\varphi(C^2) \leqslant \varphi(D^2)$.

Application of Theorem gives that φ is a nonnegative scalar multiple of the trace. □

Corollary 2. *Let* $1 < p < \infty, 0 < \lambda < \infty$, *and let* φ *be a positive linear functional on* $\mathscr{M}_n$ *such that*

$$\varphi\big((A+\lambda)^p\big) \leqslant \varphi\big((B+\lambda)^p\big), \tag{9}$$

whenever $0 \leqslant A \leqslant B$. *Then* φ *is a nonnegative scalar multiple of the trace.*

Proof. To reduce the assertion of the corollary to theorem, we use a trick from [4]. If $0 \leqslant A \leqslant B$ then $0 \leqslant tA \leqslant tB$ for every positive number t. By (9), we have $\varphi((tA+\lambda)^p) \leqslant \varphi((tB+\lambda)^p)$, hence $\varphi((A+\lambda/t)^p) \leqslant \varphi((B+\lambda/t)^p)$. Letting t to infinity we obtain $\varphi(A^p) \leqslant \varphi(B^p)$, and application of Theorem completes the proof. □

The presented results supplement a list of inequalities which characterize the trace (see [2,3,5,7,8]). Like characterizations of the trace in [3,5,7,8], the ones obtained here can be extended to the framework of operator algebras. Certainly, we have not exhausted all possible characterizations of the trace by monotonicity inequalities, and the following problem appears to be interesting.

Problem. Let f be a nondecreasing function defined on an interval S, which is not matrix monotone of order 2. Let φ be a positive linear functional on $\mathscr{M}_n$, such that $\varphi(f(A)) \leqslant \varphi(f(B))$ whenever $f(A)$, $f(B)$ are well-defined and $A \leqslant B$. Does it follow that φ is a nonnegative scalar multiple of the trace?

Remark. After the first version of the present paper had been submitted, the paper of Sano and Yatsu [6] came to our attention. In that paper, they obtained some interesting characterizations of the trace via inequalities. In particular, they proved that inequalities (7)–(9) characterize tracial property. In this connection we would like to note that the main results of our paper were announced in [1].

References

[1] A.M. Bikchentaev, A.S. Rusakov, O.E. Tikhonov, Characterization of the trace by power inequalities, in: F.G. Avkhadiev, M.M. Arslanov, A.M. Elizarov (Eds.), Proceedings of the N.I. Lobachevskii Mathematical Center, Algebra and Analysis – 2004. Reports of the International Conference, vol. 23, Kazan Mathematical Society, Kazan, 2004, pp. 45–46 (in Russian).

[2] A.M. Bikchentaev, O.E. Tikhonov, Characterization of the trace by Young's inequality, J. Inequal. Pure Appl. Math. 6 (2) (2005), article 49.

[3] L.T. Gardner, An inequality characterizes the trace, Canad. J. Math. 31 (1979) 1322–1328.

[4] G. Ji, J. Tomiyama, On characterizations of commutativity of C^*-algebras, Proc. Amer. Math. Soc. 131 (2003) 3845–3849.

[5] D. Petz, J. Zemánek, Characterizations of the trace, Linear Algebra Appl. 111 (1988) 43–52.

[6] T. Sano, T. Yatsu, Characterizations of the tracial property via inequalities, J. Inequal. Pure Appl. Math. 7 (1) (2006), article 36.

[7] A.I. Stolyarov, O.E. Tikhonov, A.N. Sherstnev, Characterization of normal traces on von Neumann algebras by inequalities for the modulus, Mat. Zametki 72 (2002) 448–454. (in Russian), English translation: Math. Notes 72 (2002), 411–416.
[8] O.E. Tikhonov, Subadditivity inequalities in von Neumann algebras and characterization of tracial functionals, Positivity 9 (2005) 259–264.